• A point \( P = (x, y) \) is a **critical point** of a function \( f(x, y) \) if
  - \( f_x(a, b) = 0 \) or does not exist, and
  - \( f_y(a, b) = 0 \) or does not exist.

• Local minima and maxima of functions occur at **[ ]**.

• Some functions don’t have global extrema, however if the domain of a continuous function \( f \) is **[ ]**, then global extrema do exist and the occur either at **[ ]** or **[ ]**.

• The **discriminant** of \( f(x, y) \) at \( P = (a, b) \) is the quantity

\[
D(a, b) = 0
\]

• **Second Derivative Test:** If \( P = (a, b) \) is a critical point of \( f(x, y) \), then
  - If \( D(a, b) > 0 \), \( f_{xx}(a, b) > 0 \), then \( (a, b) \) is a
  - If \( D(a, b) > 0 \), \( f_{xx}(a, b) < 0 \), then \( (a, b) \) is a
  - If \( D(a, b) < 0 \), then \( (a, b) \) is a
  - If \( D(a, b) = 0 \), then

• **Method of Lagrange multipliers:** The local extreme values of \( f(x, y) \) subject to a constraint \( g(x, y) = 0 \) occur at points \( P \) called critical points, satisfying the Lagrange condition **[ ]**. This condition is equivalent to the **Lagrange equations**: **[ ]**, **[ ]**.

• If the constraint curve \( g(x, y) = 0 \) is bounded then global minimum and maximum values of \( f \) subject to the constraint exist.

• A general procedure for using the method is: Write out the Lagrange equations, solve for \( \lambda \) in terms of \( x, y \), solve for \( x, y \) using the constraint, calculate the critical values.

• The Lagrange condition for a function of three variables \( f(x, y, z) \) subject to two constraints \( g(x, y, z) = 0 \) and \( h(x, y, z) = 0 \):

\[
\nabla f = 0.
\]
1. (a) Find and classify the critical points of \( f(x, y) = x^3 + 2xy - 2y^2 - 10x \).
(b) Determine the global extreme values for \( f(x, y) = x^3 + x^2y + 2y^2 \) on the domain \( x \geq 0, y \geq 0, x + y \leq 1 \).

2. Find the minimum and maximum values of \( f \) subject to given constraint.
   (a) \( f(x, y) = 2x + 3y, \quad x^2 + y^2 = 4 \)
   (b) \( f(x, y, z) = x^2 - y - z, \quad x^2 - y^2 + z = 0 \)

3. Let \( f(x, y) = (x^2 + y^2)e^{-x^2-y^2} \).
   (a) Verify that the set of critical points of \( f \) consists of the origin \((0, 0)\) and the unit circle \( x^2 + y^2 = 1 \).
   (b) Show that \( f \) takes on its maximum value on the unit circle by analyzing the function \( g(t) = te^{-t} \) for \( t > 0 \). (The Second Derivatives Test is messy here, and it fails anyways.)

4. Show that the point \( (x_0, y_0) \) closest to the origin on the line \( ax + by = c \) has coordinates
   \[
   x_0 = \frac{ac}{a^2 + b^2}, \quad y_0 = \frac{bc}{a^2 + b^2}.
   \]

5. Find the maximum value of \( f(x, y) = x^a y^b \) for \( x \geq 0, y \geq 0 \) on the line \( x + y = 1 \), where \( a, b > 0 \) are constants.
• A point \( P = (x, y) \) is a **critical point** of a function \( f(x, y) \) if
  
  - \( f_x(a, b) = 0 \) or does not exist, and
  - \( f_y(a, b) = 0 \) or does not exist.

• Local minima and maxima of functions occur at critical points.

• Some functions don’t have global extrema, however if the domain of a continuous function \( f \) is closed and bounded, then global extrema do exist and the occur either at critical interior points or in the boundary.

• The **discriminant** of \( f(x, y) \) at \( P = (a, b) \) is the quantity

\[
D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{x,y}(a, b)^2
\]

• **Second Derivative Test:** If \( P = (a, b) \) is a critical point of \( f(x, y) \), then
  
  - If \( D(a, b) > 0 \), \( f_{xx}(a, b) > 0 \), then \((a, b)\) is a **local minimum**
  - If \( D(a, b) > 0 \), \( f_{xx}(a, b) < 0 \), then \((a, b)\) is a **local maximum**
  - If \( D(a, b) < 0 \), then \((a, b)\) is a **saddle point**
  - If \( D(a, b) = 0 \), then the text is inconclusive.

• **Method of Lagrange multipliers:** The local extreme values of \( f(x, y) \) subject to a constraint \( g(x, y) = 0 \) occur at points \( P \) called critical points, satisfying the Lagrange condition \( \nabla f_P = \lambda \nabla g_P \). This condition is equivalent to the **Lagrange equations**:

\[
\begin{align*}
f_x(x, y) &= \lambda g_x(x, y), \\
f_y(x, y) &= \lambda g_y(x, y).
\end{align*}
\]

• If the constraint curve \( g(x, y) = 0 \) is bounded then global minimum and maximum values of \( f \) subject to the constraint exist.

• A general procedure for using the method is: Write out the Lagrange equations, solve for \( \lambda \) in terms of \( x, y \), solve for \( x, y \) using the constraint, calculate the critical values.

• The Lagrange condition for a function of three variables \( f(x, y, z) \) subject to two constraints \( g(x, y, z) = 0 \) and \( h(x, y, z) = 0 \):

\[
\nabla f = \lambda \nabla g + \mu \nabla h.
\]
1. (a) Find and classify the critical points of \( f(x,y) = x^3 + 2xy - 2y^2 - 10x \).
   \[ \text{Answer: Saddle at (0,0) and local minimum at (1/3,1/3).} \]

   (b) Determine the global extreme values for \( f(x,y) = x^3 + x^2y + 2y^2 \) on the domain \( x \geq 0, y \geq 0, x+y \leq 1 \).
   \[ \text{Answer: Global minimum of 0 at (0,0) and global maximum of 2 at (0,1)} \]

2. Find the minimum and maximum values of \( f \) subject to given constraint.
   (a) \( f(x, y) = 2x + 3y, \quad x^2 + y^2 = 4 \)
   \[ \text{Answer: The maximum is 26/\sqrt{13}, and the minimum is -26/\sqrt{13}.} \]

   (b) \( f(x,y,z) = x^2 - y - z, \quad x^2 - y^2 + z = 0 \)
   \[ \text{Answer: } f \text{ does not have a minimum or maximum subject to the constraints.} \]

3. Let \( f(x,y) = (x^2 + y^2)e^{-x^2-y^2} \).
   (a) Verify that the set of critical points of \( f \) consists of the origin (0,0) and the unit circle \( x^2 + y^2 = 1 \).

   (b) Show that \( f \) takes on its maximum value on the unit circle by analyzing the function \( g(t) = te^{-t} \) for \( t > 0 \). (The Second Derivatives Test is messy here, and it fails anyways.)
   \[ \text{Answer: The function } g(t) \text{ has a global maximum at } t = 1 \text{ (verify this!). Since } f(x,y) = g(x^2 + y^2), \text{ this implies that } f \text{ must have its maximum when } x^2 + y^2 = 1. \]

4. Show that the point \((x_0, y_0)\) closest to the origin on the line \( ax + by = c \) has coordinates

\[
x_0 = \frac{ac}{a^2 + b^2}, \quad y_0 = \frac{bc}{a^2 + b^2}.
\]

\[ \text{Solution: We want to minimize } d(x,y) = \sqrt{x^2 + y^2} \text{ subject to the constraint } g(x,y) = ax + by - c = 0. \text{ However, we can simplify this by noticing that } d(x,y) \text{ is exactly when } f(x,y) = d^2(x,y) = x^2 + y^2 \text{ is minimized. Thus the Lagrange equation is } \\
\langle 2x, 2y \rangle = \nabla f(x,y) = \lambda \nabla g(x,y) = \lambda \langle a, b \rangle, \]

and so we have \( 2x = \lambda a, \quad 2y = \lambda b \).

Solving each for \( \lambda \) yields \( \lambda = 2x/a \) and \( \lambda = 2y/b \), and setting these two equal yields \( y = (b/a)x \). Next, substituting this into the constraint equation yields

\[
ax + b \left( \frac{b}{a}x \right) = c, 
\]

which has solution \( x = ac/(a^2 + b^2) \). Substituting \( x = (a/b)y \) into the constraint equation gives \( y = bc/(a^2 + b^2) \), so we’re done.
5. Find the maximum value of $f(x, y) = x^a y^b$ for $x \geq 0$, $y \geq 0$ on the line $x + y = 1$, where $a, b > 0$ are constants.

**Answer:** The maximum value is

$$\frac{a^a b^b}{(a + b)^{a+b}}.$$